Recall $S' \to S$ fpqc $X'/S'$ scheme

Descent datum $/S'$ $\underset{\text{def}}{=}$ $(X', \phi)$ $\phi : S'' \underset{p_1, S'}{\times} X' \xrightarrow{\cong} S'' \underset{p_2, S'}{\times} X'$
(of schemes)

s.th. cocycle cond holds.

Effective $\underset{\text{def}}{=}$ $\exists\, X/S$ s.th. $(X', \phi) \cong (S' \underset{S}{\times} X, \text{can})$

$S, S'$ affine from now on.

Seen $(X', \phi)$ effective $\Leftarrow$ $\exists$ $\phi$ stable affine covering of $X'$

Example 1 $S' \to S$ finite étale Galois covering w/ grp $\Gamma$

$X' \to S'$ projective in sense $\exists$ immersion $X' \hookrightarrow \mathbb{P}^n_{S'}$

Then any descent datum for $X'$ effective.

Proof $x' \in X'$ any. $X'$ projective

$\to$ $\exists$ affine $U$ containing $\Gamma \cdot x'$.

Then $\bigcap_{\gamma \in \Gamma} \gamma \cdot U$ is affine ($X' \to S'$ separated)

+ $\Gamma$-stable + contains $\Gamma \cdot x'$. $\square$

Rmk Generalizes to $S' \to S$ finite loc. free.

<u>Example 2</u> $L = K(\sqrt{a})/K$ quadratic field extn, char $k \neq 2$

$E: \; y^2 = f(x) \subseteq \mathbb{P}^2_K$ EC in simplified Weierstraß form.

$[S' \to S] = [\operatorname{Spec} L \xrightarrow{p} \operatorname{Spec} K]$ Galois cover, Gal. group $\Gamma = \{1, \sigma\}$

Canonical descent datum in terms of $\Gamma$-action:

$$p^*E = (L \otimes_K E, \; \sigma^* \otimes 1)$$

$$\longleftrightarrow \quad L[x,y]/_{y^2 - f(x)} \xrightarrow{\sigma^*} L[x,y]/_{y^2 - f(x)}$$

$$\sum a_{ij} x^i y^j \longmapsto \sum \sigma(a_{ij}) x^i y^j$$

(on open $E_L \setminus e$)

Define new action by having $\sigma$ act on $[-1] \circ \sigma^*$

$$\longleftrightarrow \quad \sum a_{ij} x^i y^j \overset{\mu}{\longmapsto} \sum (-1)^j \sigma(a_{ij}) x^i y^j.$$

Descends to <u>quadratic twist</u> $\tilde{E}/K$ for $L/K$:

$$\tilde{E} \setminus \{e\} = \operatorname{Spec} \left( L[x,y]/_{y^2 - f(x)} \right)^{\mu = id}$$

$$= \operatorname{Spec} K[x, \sqrt{a} \cdot y]/_{y^2 - f(x)}$$

$$\cong \operatorname{Spec} K[x,y]/_{a y^2 - f(x)}.$$

Can it happen that $E' \cong E$?

<u>Example 3</u> (General answer)

$L/K$ Galois w/ group $\Gamma$, $X/K$ q-proj variety.

$\Longrightarrow \quad p^* X = (X_L + \text{canonical } \Gamma\text{-action})$

$$\begin{tikzcd}
X_L \arrow[d, equal] \arrow[drr, bend left, "\gamma"] & & \\
L \otimes_{\gamma, L} X_L \arrow[rr] \arrow[d] & & X_L \arrow[d] \\
\operatorname{Spec} L \arrow[rr, "\operatorname{Spec}\gamma"] & & \operatorname{Spec} L
\end{tikzcd}$$

$\Gamma$-action

canonically because $X$ comes from $K$

Let $(X_L, \Gamma \overset{\alpha}{\curvearrowright} X_L)$ be any other descent datum.

Then $\forall \gamma \in \Gamma \quad c(\gamma) := \gamma^{-1} \alpha(\gamma) \in \operatorname{Aut}(X_L/L)$

Have $\quad c(\gamma_1 \gamma_2) = \gamma_2^{-1} \gamma_1^{-1} \alpha(\gamma_1) \alpha(\gamma_2)$ $\quad (*)$

$$= \underbrace{(\gamma_2^{-1} c(\gamma_1) \gamma_2)} \cdot c(\gamma_2)$$

This comes from an action $\Gamma \curvearrowright \operatorname{Aut}(X_L/L)$

$$\gamma \bullet c := \gamma^{-1} c \gamma.$$

**Def** Cocycles $Z^1(\Gamma, A)$ $\quad A :=$ group w/ $\Gamma$-action

$= $ maps $c : \Gamma \to A$ s.t. $c(\gamma_1 \gamma_2) = (\gamma_2 \bullet c(\gamma_1)) \cdot c(\gamma_2)$

<u>Thus</u> Descent datum for $X_L$ gives

cocycle $c : \Gamma \to \mathrm{Aut}(X_L / L)$.

Conversely, if $c \in Z^1(\Gamma, \mathrm{Aut}(X_L/L))$,

then $\alpha(\gamma) := \gamma c(\gamma)$ defines descent

datum from canonical one.

$X^c :=$ descent of $(X_L, \mathrm{can} \circ c)$

Then $X^{c_1} \cong X^{c_2} \iff p^* X^{c_1} \cong p^* X^{c_2}$

fully faithfulness

i.e. $(X_L, \mathrm{can} \circ c_1) \cong (X_L, \mathrm{can} \circ c_2)$

$\iff \exists\, \beta \in \mathrm{Aut}(X_L / L)$ that intertwines two $\Gamma$-actions

i.e. $\forall\, \gamma \in \Gamma$, $\beta \gamma c_1(\gamma) = \gamma c_2(\gamma) \beta$

$\iff c_1(\gamma) = (\gamma \bullet \beta)^{-1} c_2(\gamma) \beta$

i.e. $c_1 \sim_\beta c_2$ $\Gamma$-twisted conjugate

twists or forms of $X$

Cor $\left\{ \text{varieties } \tilde{X}/K \text{ s.th. } \tilde{X}_L \cong X_L \right\} /_{\text{iso}}$

$$\overset{1:1}{=} H^1(\Gamma, \mathrm{Aut}(X_L/L)) := Z^1(\Gamma, \mathrm{Aut}(X_L/L))/_\sim$$

Application $E/K$ EC with $j \neq \{0, 1728\}$

$\Leftrightarrow \mathrm{Aut}(E_L/L) = \{\pm 1\} \quad \forall L/K$

$L/K$ Galois group $\Gamma$.

$\Gamma \subset \mathrm{Aut}(E_L/L) = \mathrm{Aut}(E/K)$ trivially.

Cocycle condition becomes $c(\gamma_1 \cdot \gamma_2) = c(\gamma_1)\, c(\gamma_2)$

+ $\Gamma$-twisted conjugation trivial as well ($\mathbb{Z}/2$ commutative)

$\Longrightarrow \{\text{forms of } E\}/_\cong \overset{1:1}{=} \mathrm{Hom}(\Gamma, \mathbb{Z}/2)$

$\overset{1:1}{=}$ Quadratic $K \underset{2}{\subset} M \subset L$.

Rmk Same applies to $\mathbb{G}_m$ since $\mathrm{Aut}(\mathbb{G}_m) = \{\pm 1\}$.

## Example 4

**Prop** $S' \to S$ fpqc. Any descent datum of elliptic curves $(E', \phi)/S'$ is effective.

**Proof** $\mathcal{L}' := \mathcal{O}_{E'}([e'])$ relatively ample.

$\phi : p_1^* E' \to p_2^* E'$ map of ECs

$\Rightarrow \quad \phi^*(p_2^*[e']) = p_1^*[e']$ in sense of equality of closed subschemes.

In other words, the natural map

$$\begin{array}{ccc} \phi^* \mathcal{O}_{p_2^* E'} & \xrightarrow{\phi^*} & \mathcal{O}_{p_1^* E'} \\ \cup & & \cup \\ \phi^* \mathcal{I}_{p_2^*[e']} & \xrightarrow{\cong} & \mathcal{I}_{p_1^*[e']} \end{array}$$

Its inverse provides a descent datum for $\mathcal{L}'$.
(Cocycle condition follows from that of $\phi$.) $\square$

**Cor** $(E',\alpha')/S'$ EC w/ level-$n$-str $n \geq 3$.

Assume $p_1^*(E',\alpha')$, $p_2^*(E',\alpha')$ are isomorphic.

Then there is a (unique) $(E,\alpha)/S$ s.th.

$$(E,\alpha)_{S'} \cong (E',\alpha').$$

**Proof** Pick any isomorphism $p_1^*(E',\alpha') \xrightarrow[\phi]{\cong} p_2^*(E',\alpha')$

Then $\phi$ is automatically a descent datum:

Namely $p_{13}^*\phi$, $p_{23}^*\phi \circ p_{12}^*\phi : S'' \times_{p_1, S'} (E',\alpha') \xrightarrow{\cong} S'' \times_{p_3, S'} (E',\alpha')$

both respect the level-structure, hence may only differ by the identity.

Now use descent for ECs. $\square$

**Funny Observation** $E_1, E_2/K$ ECs s.th. $E_{1,\bar{K}} \cong E_{2,\bar{K}}$

and $E_i[n] \cong (\mathbb{Z}/n)^{\oplus 2}$ for some $n \geq 3$.

Then $E_1 \cong E_2$. (Exercise)

# Application to $\mathcal{M}_n$ $\quad n \geq 3$

$$\begin{tikzcd} & \mathcal{M}_{3n} \arrow[dl, "q"'] \arrow[dr] & \\ Y_n = K \backslash \mathcal{M}_{3n} \arrow[rr, shift left, "\Phi"] & & \mathcal{M}_n \arrow[ll, shift left, "\Psi"] \end{tikzcd}$$

$$K = \ker\left(GL_2(\mathbb{Z}/3n) \longrightarrow GL_2(\mathbb{Z}/n)\right)$$

$y \in Y_n(S)$. $\quad n \geq 3 \Longrightarrow K \circlearrowright \mathcal{M}_{3n}$ freely

$\Longrightarrow q$ $K$-torsor

$\Longrightarrow \exists\, S' \longrightarrow S$ + $(E', \alpha) \in \mathcal{M}_{3n}(S')$

with $q(E', \alpha) = y$.

Then $p_1^*(E', \alpha)$, $p_2^*(E', \alpha) / S'' = S' \times_S S'$ map to same point in $Y_n$.

Freeness of $K \circlearrowright \mathcal{M}_{3n}$ implies $\mathcal{M}_{3n} \times_{Y_n} \mathcal{M}_{3n} \xleftarrow{\cong} K \times \mathcal{M}_{3n}$,

so $\exists!\, k \in K(S'')$ s.th. $p_1^*(E', \alpha) \cong p_2^*(E', \alpha \circ k)$

Multiplying level by 3, find $\phi : p_1^*(E', 3\alpha) \xrightarrow[\phi]{\cong} p_2^*(E', 3\alpha)$

Since $n \geq 3$, pair $(E, 3\alpha)$ descends to $(E, \beta)/S$

Put $\Phi(y) = (E, \beta) \in \mathcal{M}_n(S)$.

Converse: $(E, \beta) \in \mathcal{M}_n(S)$.

$$\exists\, S' \to S \quad + \quad \alpha : (\mathbb{Z}/3n)^{\oplus 2} \longrightarrow E' = S' \times_S E$$

$$\text{s.t. } 3\alpha = \beta$$

Obtain $q(E', \alpha) \in Y_n(S')$.

Fully faithfulness of descent: $Y_n(S) = \mathrm{Eq}\left[ Y_n(S') \rightrightarrows Y_n(S'') \right]$

Now $p_1^*(E', 3\alpha) \cong p_2^*(E', 3\alpha)$

so $p_1^*(E', \alpha) \cong p_2^*(E', \alpha \circ k)$

for unique $k \in K(S'')$ by freeness of $K$-action.

$\Longrightarrow$ $q(E', \alpha)$ lies in Equaliser since $q$ $K$-equivariant.

This achieves the proof of representability of $\mathcal{M}_n / \mathbb{Z}[\frac{1}{n}]$

## Example 5

1) Consider relative curves of genus $g \neq 1$:

$C \to S$ proper smooth geom conn fibers of genus $g$.

Any such has canonical ample line bundle:

$\Omega^1_{C/S}$ $(g \geq 2)$ or $(\Omega^1_{C/S})^{-1}$ $(g = 0)$.

$\Rightarrow$ Any fpqc descent datum of such effective

2) (Much less trivial) Consider $(A, \lambda)$

$A/S$ projective abelian scheme

$\lambda : A \to A^\vee$ ($\Rightarrow A^\vee$ representable, again projective)

polarization $\overset{=}{\text{def}}$ $\exists$ étale surjective $S' \to S$

+ ample line bundle $L'$ on $A_{S'}$ s.th. $\lambda_{S'} = \phi_{L'}$.

Then such pairs $(A, \lambda)$ satisfy fpqc descent.

(Abelian schemes themselves probably do not satisfy fpqc descent.)

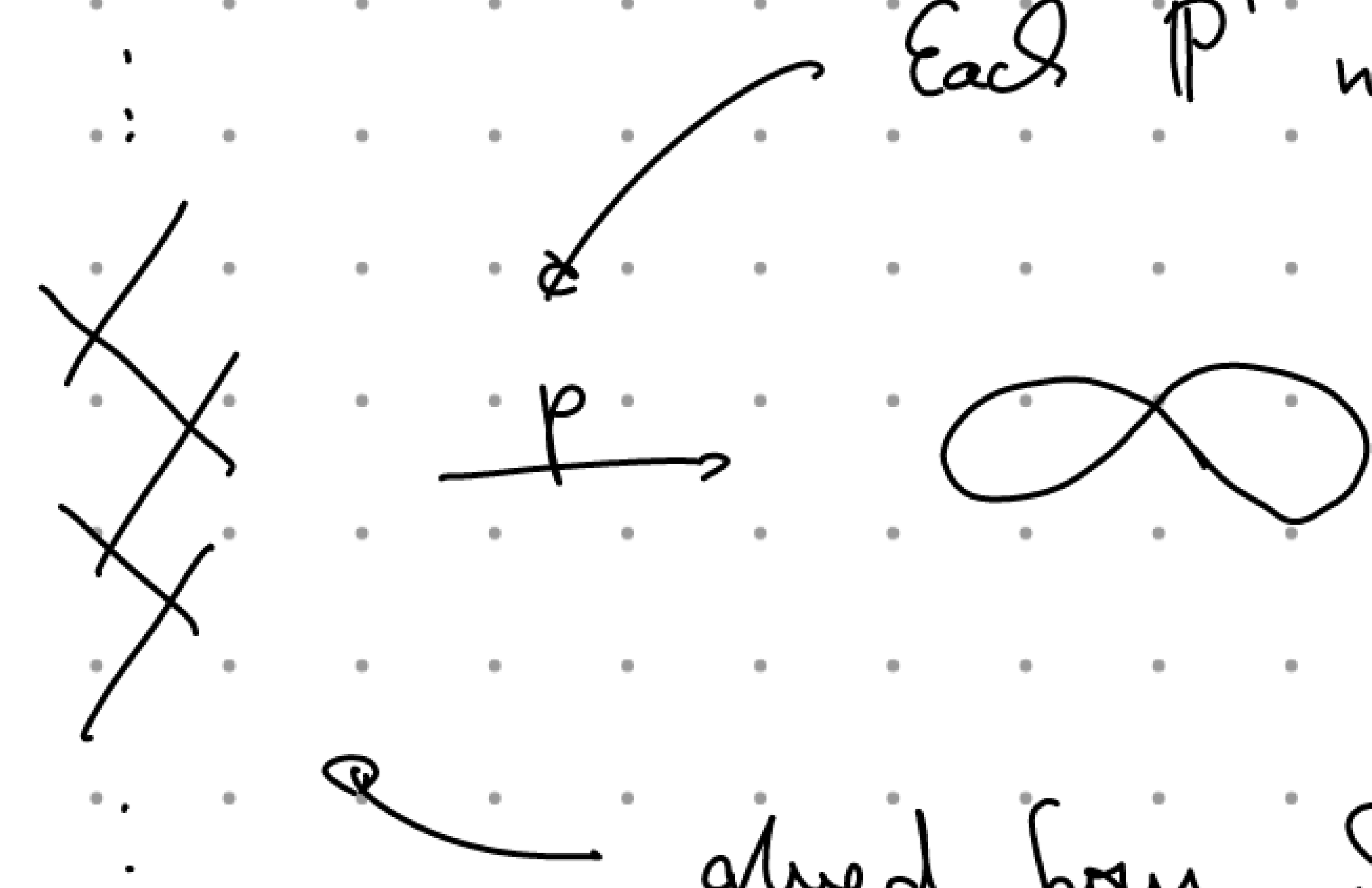
Example 6 [Raynaud, Faisceaux amples sur ...]
(also cf. Master thesis of Wouter Zomervrucht)

$S$ = Genus 0 curve w/ nodal singularity $/k$

e.g. $y^2 = x^2(x-1) \subset \mathbb{P}^2_k$.

$S' \longrightarrow S$ Galois cover w/ Gal. grp. $\mathbb{Z}$

Each $\mathbb{P}^1$ maps as normalization to $S$



$p$

glued from $\operatorname{Spec} k[x,y]/_{xy}$
union of $\mathbb{P}^1_k$'s

Local equation of $p$ on nodes

$$k[x,y]/_{y^2 - x^2(x+1)} \longrightarrow k[s,t]/_{t^2 - (s^2-1)^2}$$

$$x \longmapsto s^2 - 1$$

$$y \longmapsto st$$

Unramified above $x = y = 0 \Rightarrow t = 0,\ s = \pm 1$

$dx \mapsto 2s\,ds$ , $dy \mapsto s\,dt + \cancel{t\,ds}^{0}$

$s$ unit above $x=y=0$ $\longrightarrow$ Unramified.

Let $E_0/k$ EC, $x \in E(k)$ infinite order.

$E := S \times_{\operatorname{Spec} k} E_0$ , $E' = S' \times_{p,S} E$

$\operatorname{Hom}(\mathbb{P}^1_k, E_k) = 0$ by Riemann-Hurwitz,

so $E(S) = E'(S') = E_0(k)$

<u>Descent datum</u> $\mathbb{Z} \circlearrowright E'$ via $\left\{ \begin{array}{l} n \text{ acts as translation on } S' \\ \& \ + nx \text{ on fibers} \end{array} \right.$

Raynaud XIII.3.1 : Is effective, descends to

curve of genus 1 $C/S$.

Then $C/S$ is a <u>non-projective</u> smooth proper family of genus 1 curves.

(Intuitively: $x$ having infinite order

$\implies$ there is no $\mathbb{Z}$-stable horizontal divisor

on $E'$.)

<u>Note</u> $\exists$ action $E \times_S C \longrightarrow C$ making $C$ into $E$-torsor.

<u>Prop</u> [Raynaud XIII Prop 2.6] $S$ normal, local

$A/S$ abelian scheme $S' \longrightarrow S$ étale covering

$(A_{S'}, \phi)$ descent datum where $\phi$ is given

as $t_x \circ \mathrm{can}$ for $x \in A(S'')$.

Then $(A_{S'}, \phi)$ effective $\Leftrightarrow$ Class of $\phi$ in $H^1(S', A)$ torsion.